One can check in Eq. (14) that no root lies outside the unit circle provided $(k/m)\Delta t^2 \leq 10$. Thus, the algorithm is stable and becomes unstable only in the high-frequency range. The proposed algorithm bears also attractive properties of much greater accuracy.[8,11]

Discussion

It was shown that the straightforward application of the Ritz procedure with piecewise cubic Hermitian interpolation functions in order to produce time finite element equations will result in an unstable algorithm. This is a rather disappointing result, if one refers, e.g., to the second-order static equilibrium problem of a rod for which the same interpolation model works so well. In the authors' opinion, in order to get a better understanding of this phenomenon, one must refer to the parent matrix K before the initial or the boundary conditions are imposed.

The degeneracy of K represents the need of the number of the essential conditions that must be imposed in order to make the formulation well defined for the given problem. The parent matrix of Eqs. (4) and (5) is singular of degeneracy 1. This is exactly the single rigid-body motion that must be removed in order to make the formulation for the "static" rod problem well defined. However, in the dynamic problem, both the initial displacement and velocity are essential conditions that must be imposed in order to make the formulation well defined. Indeed, the parent matrix of the modified elements [Eqs. (12) and (13)] is singular of degeneracy 2 and both the initial displacement and velocity must be imposed in order to solve the equations.

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A Quasivariational Principle for Fluid-Solid Interaction

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Introduction

In mechanics, the governing equations are often expressed by Euler equations of variational principles, which are formulated either through a general principle of mechanics or a trial-and-error method. The classical (integral) type of variational principles (e.g., Hamilton's principle) admits an explicit functional, while the quasitype (differential type) of variational principles (e.g., D'Alembert's principle and the principle of virtual work or power) denies it. Only quasivariational principles may be contrived for viscous fluids,[1] and, hence, for their strong interaction with elastic solids.

The fluid-solid strong interaction is of primary concern in hydro- and acousto-elastic vibrations, and it requires a simultaneous analysis of the fluid-solid field at high frequencies.[2,3] In the analysis, the governing equations of combined fluid-solid field, including the interface conditions, can be expressed by a quasivariational principle; this, in turn, provides a unified direct method of approximate calculations for their solutions. Accordingly, variational principles were derived, though only a few (e.g., Refs. 4-6 and references therein), for the fluid-solid field by paralleling to those for fluids and solids.[7] In these principles, the interface conditions were put aside as constraints or incorporated into each principle, although their validity may be straightforward, by a purely formal manner and without any clear discussion. Thus, the aim of this Note is to derive systematically a quasivariational principle which governs the strong interaction of a viscous incompressible fluid and a linear elastic solid immersed within the fluid of finite extent.

Interface Equations

At a certain time $t=t_0$, consider a regular region of viscous incompressible fluid $\Omega+\partial\Omega$ and one of a linear elastic solid $B+\partial B$, which are referred to by a fixed, right-handed system of Cartesian coordinates x_i in the Euclidean three-space Ξ. The viscous fluid with no surface waves undergoes only small amplitude vibrations, as does the elastic solid; hence, the motions of both media are governed by the linearized equations. The solid region is immersed within the fluid region, and at their interface ∂B, no cavitation is considered, allowing these regions to adhere to each other on this surface. Thus, at the interface

$$\underline{t}+\underline{\sigma}=0,\ \underline{v}-\dot{\underline{u}}=0 \text{ on } \partial B\times T=[t_0,t_1) \tag{1}$$

where the superimposed dot indicates time differentiations; $\underline{v}$ and $\underline{t}$ are the velocity and stress vectors of fluid; $\underline{u}$ and $\underline{\sigma}$ denote the displacement and stress vectors of solid. The first

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of Eqs. (1) may also be written in terms of the symmetric components of stress tensor of fluid t_{ij} and those of solid σ_{ij} as

$$\nu_i(t_{ij}-\sigma_{ij})=0 \text{ on } \partial B\times T \tag{2}$$

with

$$t_j=n_i t_{ij} \text{ in } \bar{\Omega}\times T,\ \sigma_j=\nu_i\sigma_{ij} \text{ in } \bar{B}\times T \tag{3}$$

where the closures of regions are denoted by $\bar{\Omega}(=\Omega U\partial\Omega)$ and $\bar{B}(=BU\partial B)$; $\underline{n}$ and $\underline{\nu}$ are the unit outward vectors normal to $\partial\Omega$ and ∂B, respectively. In the above equations and henceforth, standard index notation is employed; accordingly, Einstein's summation convention is implied for repeated Latin (1, 2, 3) and Greek indices (1, 2).

Principle of Virtual Power

In order to derive the quasivariational principle of fluid-solid strong interaction, as a starting point, the principle of virtual power is taken for the viscous fluid, and the principle of virtual work for the elastic solid.[8,9] In deriving the variational principles of fluids, the principle of virtual power is more suitable than Hamilton's principle (cf., Refs. 10 and 11), and is also a basis for finite element modeling in fluid mechanics,[12] as is its counterpart, the principle of virtual work in solid mechanics. In the region $\Omega+\partial\Omega$ fixed in space Ξ, the principle of virtual power is expressed by

$$\int_\Omega t_{ij}\delta d_{ij}\mathrm{d}V=\int_\Omega \rho(f_i-a_i)\delta v_i\mathrm{d}V+\int_{\partial\Omega} t_i^*\delta v_i\mathrm{d}S \tag{4}$$

where $\underline{f}$ is the body force vector per unit mass of fluid with its mass density ρ, $\underline{a}$ the acceleration vector; d_{ij} is the rate of deformation (velocity strain) tensor defined in the form

$$d_{ij}=\tfrac{1}{2}(v_{i,j}+v_{j,i}) \text{ in } \bar{\Omega}\times T \tag{5}$$

where an asterisk stands for a prescribed quantity and a comma for partial differentiations with respect to x_i.

Variational Principle for Viscous Fluid

By substitution of the velocity strain expression Eq. (5) into Eq. (4) and integrating over the time interval T, there results the variational equation in a Euclidean coordinate system

$$\int_T \mathrm{d}t\int_\Omega [\tau_{ij,i}-p_{,j}+\rho(f_j-a_j)]\delta v_j\mathrm{d}V$$
$$-\int_T \mathrm{d}t\int_\Omega [(-p\delta_{ij}+\tau_{ij})\delta v_j]_{,i}\mathrm{d}V+\int_T \mathrm{d}t\int_{\partial\Omega} t_i^*\delta v_i\mathrm{d}S=0 \tag{6}$$

where p is the hydrodynamic pressure, δ_{ij} Kronecker's delta, and τ_{ij} is the viscous stress tensor defined by

$$t_{ij}=-p\delta_{ij}+\tau_{ij} \text{ in } \bar{\Omega}\times T$$

Besides, the operations of variation are taken to commute with those of differentiation, and the admissible states are prescribed at $t=t_0$ and $t=t_1$ in Eq. (6). Further, by applying the general form of Green-Gauss integral transformation for the fluid region $\Omega+\partial\Omega$ with a surface of discontinuity ∂B from Eq. (6), it follows a quasivariational principle in the form

$$\delta J\langle v_i\rangle=\delta J_{ii}=0 \tag{7a}$$

with

$$\delta J_{11}=\int_T \mathrm{d}t\int_\Omega [\tau_{ij,i}-\delta_{ij}p_{,i}+\rho(f_j-a_j)]\delta v_j\mathrm{d}V$$

$$\delta J_{22}=\int_T \mathrm{d}t\int_{\partial\Omega\cap\partial B} [t_j^*-n_i(-\delta_{ij}p+\tau_{ij})]\delta v_j\mathrm{d}S$$

$$\delta J_{33}=\int_T \mathrm{d}t\int_{\partial B} \nu_i[(-\delta_{ij}p+\tau_{ij})\delta v_j]\mathrm{d}S \tag{7b}$$

where boldface brackets indicate a jump of the enclosed quantity χ across the interface; i.e., $\chi=\chi_{\text{fluid}}-\chi_{\text{solid}}$. Equation (7) reduces to a familiar form of the variational principle of viscous incompressible fluid if the fluid contains no solid within itself; that is, δJ_{33} is dropped out. This recovers the principle given for a constrained body where virtual displacements were replaced by virtual velocities.[8]

Principle of Virtual Work (An Associated Variational Principle for Elastic Solids)

The principle of virtual work asserts that

$$\int_B \sigma_{ij}\delta e_{ij}\mathrm{d}V=\int_B q(g_j-b_j)\delta u_j\mathrm{d}V+\int_{\partial B}\sigma_i^*\delta u_i\mathrm{d}S \tag{8}$$

Here, $\boldsymbol{g}$ is the body force vector per unit mass of elastic solid with its mass density q, $\boldsymbol{b}(=\ddot{\boldsymbol{u}})$ the acceleration vector, and e_{ij} is the linear strain tensor defined by

$$e_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}) \text{ in } \bar{B}\times T \tag{9}$$

As before, by inserting Eq. (9) into Eq. (8) and carrying out the appropriate operations, a one-field, quasivariational principle of elastic solid is obtained as

$$\delta I\langle u_i\rangle=\delta I_{\alpha\alpha}=0 \tag{10a}$$

with

$$\delta I_{11}=\int_T \mathrm{d}t\int_B [\sigma_{ij,i}+q(g_j-b_j)]\delta u_j\mathrm{d}V$$

$$\delta I_{22}=\int_T \mathrm{d}t\int_{\partial B}(\sigma_j^*-\nu_i\sigma_{ij})\delta u_j\mathrm{d}S \tag{10b}$$

Quasivariational Principle for Viscous Fluid-Elastic Solid Strong Interaction

To begin, a combined two-field, quasivariational principle for the region $\Omega+\partial\Omega$ is obtained by summing δJ and δI of Eqs. (7) and (10) as

$$\delta\Gamma\langle u_i,v_i\rangle=\delta J_{ii}+\delta I_{\alpha\alpha}=0 \tag{11}$$

and by imposing the usual continuity conditions of velocity and equilibrium equations of traction at the interface, Eqs. (2) and (3). The quasivariational principle (11) generates, as its Euler equations, the equations of motion and the natural traction boundary conditions for both the fluid and solid media. Now, to incorporate the constraint conditions of Eqs. (2) and (3) into Eq. (11), the quasivariational principle is modified by adding a dislocation potential Δ each constraint as a zero times a Lagrange multiplier λ (Friedrichs's transformation) as

$$\delta L\langle u_i,\ v_i\rangle=\delta\Gamma+\delta\Delta=0 \tag{12a}$$

with

$$\Delta=\int_T \mathrm{d}t\int_{\partial B}\lambda_i(v_i-\dot{u}_i)\mathrm{d}S \tag{12b}$$

In Eqs. (12), the indicated variations of field variables as well as those of λ are to be varied as free (arbitrary and independent). By performing all of the variations, these equations take the form

$$\delta L\langle u_i,v_i\rangle=\delta J_{\alpha\alpha}+\delta I_{\alpha\alpha}+\delta D=0 \tag{13a}$$

with

$$\delta D=\int_T \mathrm{d}t\int_{\partial B}\{\delta\lambda_i(v_i-\dot{u}_i)+\lambda_i(\delta v_i-\delta\dot{u}_i)+\nu_i[t_{ij}\delta v_j]\}\mathrm{d}S \tag{13b}$$

From Eq. (13), it follows that

$$\lambda_j + \nu_i(-\delta_{ij}p + \tau_{ij}) = 0, \quad \lambda_j + \nu_i\sigma_{ij} = 0 \tag{14}$$

By solving these equations for λ, one finds the appropriate form to be used in Eq. (12) as

$$\lambda_j = -\tfrac{1}{2}\nu_i(-\delta_{ij}p + \tau_{ij} + \sigma_{ij}) \tag{15}$$

After substituting Eq. (15) into Eq. (13), the first variation δD is expressed by

$$\delta D = \delta D_{\alpha\alpha} \tag{16a}$$

with

$$\delta D_{11} = \int_T \mathrm{d}t \int_{\partial B} \tfrac{1}{2}\nu_i(-\delta_{ij}\delta p + \delta\tau_{ij} + \delta\sigma_{ij})(\dot{u}_j - v_j)\,\mathrm{d}S$$

$$\delta D_{22} = \int_T \mathrm{d}t \int_{\partial B} \tfrac{1}{2}\nu_i[(-\delta_{ij}p + \tau_{ij}) - \sigma_{ij}](\delta\dot{u}_j + \delta v_j)\,\mathrm{d}S \tag{16b}$$

This takes into account all the interface equations in Eq. (12); hence, a unified, quasivariational principle is concluded as follows.

Quasivariational Principle

Let $\Omega + \partial\Omega$ and $B + \partial B$ denote regular finite and bounded regions with their sufficiently smooth boundary surfaces $\partial\Omega$ and ∂B, and their closures $\bar{\Omega}$ and $\bar{B}$ in the Euclidean three-space Ξ. Also, let the elastic anisotropic solid $B + \partial B$ be immersed within the region of viscous, incompressible, nonpolar fluid $\Omega + \partial\Omega$. Then, of all the admissible states $\Lambda = \{v_i, p, \tau_{ij}; u_i, \sigma_{ij}\}$ which satisfy 1) the equation of continuity, 2) the constitutive equations (Newtonian or Stokosian), linear or nonlinear, 3) the kinematic relations and the velocity boundary conditions for the fluid, 4) the strain-displacement relations for the solid as well as the initial conditions, and 5) the symmetry of stress tensors for both the fluid and solid; "if and only if," that admissible state Λ which satisfies i) the Euler equations of motion, ii) the natural traction boundary conditions for the viscous incompressible fluid, iii) the equations of motion for the elastic solid as well as the natural continuity conditions of velocity, and iv) equilibrium equations of traction at the interface ∂B, is determined by the variational equation in the form

$$\begin{aligned}
\delta L\langle\Lambda\rangle = \int_T \mathrm{d}t \{ &\int_\Omega [\tau_{ij,i} - p_{,j} + \rho(f_j - a_j)]\delta v_j \mathrm{d}V \\
&+ \int_{\partial\Omega \cap \partial B} [t_j^* - n_i(-\delta_{ij}p + \tau_{ij})]\delta v_j \mathrm{d}S \\
&+ \int_B [\sigma_{ij,i} + q(g_j - b_j)]\delta u_j \mathrm{d}V \\
&+ \int_{\Omega B} \tfrac{1}{2}\nu_i[(-\delta_{ij}p + \tau_{ij} - \sigma_{ij})(\delta\dot{u}_j + \delta v_j) \\
&+ (\dot{u}_j - v_j)(-\delta_{ij}\delta p + \delta\tau_{ij} + \delta\sigma_{ij})]\mathrm{d}S\} = 0
\end{aligned} \tag{17}$$

as the appropriate Euler equations.

The quasivariational principle, Eq. (17), is believed to be new, and it does agree with, and contains some of, the earlier formulations as special cases. By using the fundamental lemma of the calculus of variations, the principle leads readily to: the Euler equations of motion, the natural traction boundary conditions for the viscous fluid, the equations of motion for the elastic solid, and the natural continuity conditions of velocity and equilibrium equations of traction at the interface. Conversely, if these equations are satisfied, the principle is evidently satisfied.

The inclusion of constraint conditions, Eqs. (2) and (3), to the quasivariational principle, Eq. (12) through Friedrichs's transformation is a classical one.[13-15] However, there is a slight difference in Eq. (12) from the classical one; that is, the constraint conditions are boundary constraints; λ of Eq. (12) is a function at the interface ∂B, while the constraints usually treated are either domain or isoparametric ones.

Conclusions

In the paper, a new, unified, quasivariational principle is derived for the strong interaction of a viscous incompressible fluid of finite extent and an elastic solid immersed within the fluid, and, hence, a standard basis is provided in obtaining direct approximate solutions for the coupled vibrations of both the media at high frequencies.

The admissible state Λ of the principle should meet the rest of the fundamental equations of both the media as constraints. However, by further application of Friedrichs's transformation, the remaining equations may be similarly included in the principle. This and certain applications are in progress and will be reported later.

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